On the Entropy of Four-Dimensional Near-Extremal $N = 2$ Black Holes with R^2-Terms

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Abstract

We consider the entropy of four-dimensional near-extremal $N = 2$ black holes. Without R^2-terms, the Bekenstein-Hawking entropy formula has the structure of the extremal black holes entropy with a shift of the charges depending on the non-extremality parameter and the moduli at infinity. We construct a class of small near-extremal horizon solutions with R^2-terms. In this case the generalized Wald entropy is the same as in the extremal case, without a shift.

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1 Introduction

Explaining the microscopic origin of black holes entropy is one of the important tasks of any theory of quantum gravity. Much progress towards achieving this goal in the framework of string theory has been obtained for supersymmetric charged black holes in various dimensions. One such class of extremal black holes characterized by electric and magnetic charges (q_I, p^I) exists in compactification of type II string theory on Calabi-Yau 3-folds (CY_3) (for a review see [1, 2]). These black holes are obtained by wrapping D-branes around cycles in CY_3. Their near horizon geometry is $AdS_2 \times S^2 \times CY_3$, where the moduli of the Calabi-Yau 3-fold are fixed on the horizon by the attractor equation in terms of the charges. Recently much work has been done on extremal non-supersymmetric black holes [3].

The four-dimensional low energy effective action of type II strings compactified on CY_3 is given by $N = 2$ Poincaré supergravity coupled to $N = 2$ Abelian vector multiplets. The macroscopic extremal black holes are asymptotically flat charged supersymmetric solutions of the field equations. At leading order in the curvature, the entropy of the black holes is given by the Bekenstein-Hawking area law $S = \frac{A}{4}$, where A is the area of the event horizon and is determined in terms of the charges by the attractor mechanism. With subleading R^2-terms included, the entropy of these macroscopic black holes has been computed using the generalized entropy formula of Wald [4].

With one electric charge q_0 and p^A magnetic charges [1] one gets the entropy [5]

$$S = 2\pi \sqrt{q_0 \left(D_{ABC} p^A p^B p^C + 256 D_A p^A \right)} , \tag{1}$$

where D_{ABC} and D_A are respectively proportional to the triple intersection numbers and the second Chern class numbers of the CY_3. The first term in (1) is the Bekenstein-Hawking area law, while the second term is the R^2 generalized entropy formula correction.

The aim of this paper is to study near-extremal $N = 2$ black hole solutions and their entropy with R^2-terms included. These are non-supersymmetric solutions, the horizon is no longer $AdS_2 \times S^2$ and the attractor mechanism no longer works. The moduli of the CY_3 are not fixed at the horizon in terms of the charges and the entropy may depend on the asymptotic values of the moduli. However, when considering the Bekenstein-Hawking entropy without R^2-terms, one notices that it has the same structure as that of the extremal black holes, with the charges being shifted [6, 7]

$$q_0 \rightarrow q_0 + \frac{1}{2}\mu h_0, \qquad p^A \rightarrow p^A + \frac{1}{2}\mu h^A , \tag{2}$$

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[1]We consider type II compactification with $A = 1 \ldots b_2$ and b_2 is the second Betti number of CY_3.

where μ is the non-extremality parameter and h_0, h^A correspond to the asymptotic values of the moduli. A natural question to ask is whether this property of the near-extremal Bekenstein-Hawking entropy holds with the R^2-corrected black hole entropy (1). We construct horizon solutions for a class of near-extremal black holes with $D_{ABC}p^Ap^Bp^C = 0$, i.e. having an extremal limit with a vanishing classical horizon area. For these black holes the entropy is the same as in the extremal case, without a shift in the charges.

Note that the R^2-terms considered in this paper are F-terms. One generally expects also D-term corrections, which are not taken into account here. For supersymmetric black holes, it is conjectured that such terms do not contribute to the entropy [8].

The paper is organized as follows: In section 2 we give a brief review of four-dimensional $N = 2$ supergravity with R^2-terms. In section 3 we derive a generalized Wald entropy formula for non-extremal $N = 2$ black holes with R^2-terms. In section 4 we present the horizon solution of near-extremal $N = 2$ black hole and compute the entropy. We will first review previous results without R^2-terms, and then present the new results with R^2-terms.

In the paper we will use $a, b, \ldots = 0, 1, 2, 3$ to denote the tangent space indices, corresponding to the indices $\mu, \nu, \ldots$ of the space-time coordinates (t, r, ϕ, θ). The exception are $i, j = 1, 2$ which are gauge $SU(2)$ indices, and $\alpha = 1, 2$ which is a global $SU(2)$ index. The sign conventions for the curvature tensors follow [1].

2 R^2-Terms in $N = 2$ Supergravity - A Brief Review

We will consider $N = 2$ Poincaré supergravity coupled to N_V Abelian $N = 2$ vector multiplets. $N = 2$ Poincaré supergravity is a supersymmetric extension of Einstein-Maxwell gravity, adding two spin 3/2 gravitini to the graviton and (gravi-)photon. $N = 2$ Poincaré supergravity can be formulated as a gauge fixed version of $N = 2$ conformal supergravity coupled to an $N = 2$ Abelian vector multiplet (see [1] for a comprehensive review).

The on-shell field content of the vector multiplet is a complex scalar, a doublet of Weyl fermions, and a vector gauge field. We will consider $N_V + 1$ vector multiplets, and will denote by X^I, $I = 0 \ldots N_V$, the scalars (moduli) in the vector multiplets. The couplings of the vector multiplets are encoded in a prepotential $F(X^I)$, which is a homogenous of second degree holomorphic function.

The $N = 2$ conformal supergravity multiplet (Weyl multiplet) is denoted by W^{abij}. It consists of gauge fields for the local symmetries: translations (P), Lorentz transformations (M),

dilatations (D), special conformal transformations (K), $U(1)$ transformations (A), $SU(2)$ transformations (V), and supertransformations (Q, S). In the theory without R^2-terms, the Weyl multiplet appears in the Lagrangian through the superconformal covariantizations. In order to get the R^2-terms, one adds explicit couplings to the Weyl multiplet. This appears in the form of a background chiral multiplet, which is equal to the square of the Weyl multiplet W^2. The lowest component of the chiral multiplet is a complex scalar denoted $\widehat{A}$. The prepotential $F(X^I, \widehat{A})$ describes the coupling of the vector multiplets and the chiral multiplet.

We consider a prepotential of the form

$$F = \frac{D_{ABC} X^A X^B X^C}{X^0} + \frac{D_A X^A}{X^0} \widehat{A} \, , \tag{3}$$

where D_{ABC}, D_A are constants and $A, B, C = 1 \ldots N_V$. This prepotential arises, for instance, from a compactification of type IIA string theory on a Calabi-Yau three-fold. The coefficients in the prepotential are topological data of the Calabi-Yau three-fold: $-6D_{ABC}$ are the triple intersection numbers (symmetric in all indices), and $-1536D_A$ are the second Chern class numbers. The first term in the prepotential arises at tree-level in α' and in g_s. The second term arises at tree-level in α' and is at one-loop in g_s. It describes R^2 couplings in the Lagrangian. In the large Calabi-Yau volume approximation $\text{Im}(X^A/X^0) \gg 1$, all other corrections are suppressed and the prepotential consists of only these two terms. We will assume this approximation to be valid near the horizon by an appropriate hierarchy of charges. One introduces the notation:

$$F_I \equiv \frac{\partial}{\partial X^I} F(X^I, \widehat{A}), \qquad F_{\widehat{A}} \equiv \frac{\partial}{\partial \widehat{A}} F(X^I, \widehat{A}) \, , \tag{4}$$

and similarly for higher order and mixed derivatives.

The bosonic part of the $N = 2$ conformal supergravity Lagrangian is

$$
\begin{aligned}
8\pi e^{-1} \mathcal{L} \;=\; & -\frac{i}{2}(\bar{X}^I F_I - X^I \bar{F}_I) R + \\
& + \Big(i\mathcal{D}^a \bar{X}^I \mathcal{D}_a F_I + \frac{i}{4} F_{IJ}(F_{ab}^{-I} - \frac{1}{4}\bar{X}^I T_{ab}^-)(F^{ab-J} - \frac{1}{4}\bar{X}^J T^{ab-}) + \\
& + \frac{i}{8} \bar{F}_I (F_{ab}^{-I} - \frac{1}{4}\bar{X}^I T_{ab}^-) T^{ab-} + \frac{i}{32} \bar{F} T_{ab}^- T^{ab-} - \frac{i}{8} F_{IJ} Y_{ij}^I Y^{ijJ} + \\
& - \frac{i}{8} F_{\widehat{A}\widehat{A}}(\widehat{B}_{ij}\widehat{B}^{ij} - 2\widehat{F}_{ab}^-\widehat{F}^{ab-}) + \frac{i}{2}\widehat{F}^{ab-} F_{\widehat{A}I}(F_{ab}^{-I} - \frac{1}{4}\bar{X}^I T_{ab}^-) + \\
& - \frac{i}{4}\widehat{B}_{ij} F_{\widehat{A}I} Y^{ijI} + \frac{i}{2} F_{\widehat{A}}\widehat{C} + \text{h.c.} \Big) + \\
& + i(\bar{X}^I F_I - X^I \bar{F}_I)\Big(\mathcal{D}^a V_a - \frac{1}{2}V^a V_a - \frac{1}{4}M_{ij}\bar{M}^{ij} + D^a \Phi^i_{\;\alpha} D_a \Phi^\alpha_{\;i}\Big) \, .
\end{aligned}
\tag{5}
$$

$e \equiv \sqrt{|\det(g_{\mu\nu})|}$ where $g_{\mu\nu}$ is the curved metric, R is the Ricci scalar, D_a is the covariant derivative with respect to all superconformal transformations, $\mathcal{D}_a$ is the covariant derivative with respect to P, M, D, A, V-transformations, F_{ab}^{-I} is the anti-selfdual part of the vector field strength, T_{ab}^- is an anti-selfdual antisymmetric auxiliary field of the Weyl multiplet, Y_{ij}^I are real $SU(2)$ triplets of auxiliary scalars of the vector multiplet, and $V_a, M_{ij}, \Phi^i_{\ \alpha}$ are components of a compensating nonlinear multiplet. The hatted fields are components of the chiral multiplet W^2, with their bosonic parts given by

$$
\begin{aligned}
\theta^0 \qquad \widehat{A} &= T_{ab}^- T^{ab-} \\
\theta^2 \qquad \widehat{B}_{ij} &= -16 R(V)_{(ij)ab} T^{ab-} \\
\widehat{F}^{ab-} &= -16 \mathcal{R}(M)_{cd}^{\ \ ab} T^{cd-} \\
\theta^4 \qquad \widehat{C} &= 64 \mathcal{R}(M)_{cd}^{\ \ ab-} \mathcal{R}(M)^{cd-}_{\ \ ab} + 32 R(V)_{ab\ j}^{\ \ i-} R(V)^{abj-}_{\ \ \ i} - 16 T^{ab-} D_a D^c T_{cb}^+ \ .
\end{aligned}
$$

$$(6)$$

$T_{ab}^+ = \bar{T}_{ab}^-$ is the selfdual counterpart of the auxiliary field, $R(V)_{ab\ j}^{\ \ i}$ is the field strength of the $SU(2)$ transformations, $\mathcal{R}(M)_{ab}^{\ \ cd}$ is the modified Riemann curvature and $\mathcal{R}(M)_{ab}^{\ \ cd-}$ is the anti-selfdual projection in both pairs of indices. The bosonic part of $\mathcal{R}(M)_{ab}^{\ \ cd}$ is given by[2]

$$
\mathcal{R}(M)_{ab}^{\ \ cd} = R_{ab}^{\ \ cd} - 4 f_{[a}^{\ [c} \delta_{b]}^{d]} + \frac{1}{32}(T_{ab}^- T^{cd+} + T_{ab}^+ T^{cd-}) \ ,
$$

$$(7)$$

where $R_{ab}^{\ \ cd}$ is the Riemann tensor, and $f_a^{\ c}$ is the connection of the special conformal transformations, determined by the conformal supergravity conventional constraints, with the bosonic part[2]:

$$
f_a^{\ c} = \frac{1}{2} R_a^{\ c} - \frac{1}{4}(D + \frac{1}{3}R)\delta_a^c + \frac{1}{2}{}^\star R(A)_a^{\ c} + \frac{1}{32} T_{ab}^- T^{cb+} \ ,
$$

$$(8)$$

where $R_a^{\ c}$ is the Ricci tensor, D is an auxiliary real scalar field of the Weyl multiplet, and ${}^\star R(A)^a_{\ b}$ is the Hodge dual of the field strength of the $U(1)$ transformations. Note that the T^2-terms in $\mathcal{R}(M)_{ab}^{\ \ cd}$ cancel exactly the T^2 contribution from $f_a^{\ c}$.

The auxiliary field D is constrained by a constraint on the nonlinear multiplet:

$$
\mathcal{D}^a V_a - D - \frac{1}{3}R - \frac{1}{2}V^a V_a - \frac{1}{4}M_{ij}\bar{M}^{ij} + D^a \Phi^i_{\ \alpha} D_a \Phi^\alpha_{\ i} = 0 \ ,
$$

$$(9)$$

where we have assumed a bosonic solution.

In order to obtain Poincaré supergravity one gauge fixes the bosonic fields (in addition there

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[2]We have assumed the K-gauge fixing which will be defined later (10).

is a gauge fixing of fermionic fields):

$$
\begin{aligned}
K-\text{gauge}: \quad & b_a = 0 \\
D-\text{gauge}: \quad & i(\bar{X}^I F_I - X^I \bar{F}_I) = 1 \\
A-\text{gauge}: \quad & X^0 = \bar{X}^0 > 0 \\
V-\text{gauge}: \quad & \Phi^i{}_\alpha = \delta^i_\alpha \ ,
\end{aligned}
\tag{10}
$$

where b_a is the connection of the dilatations.

3 Entropy Formula with R^2-Terms

With the addition of R^2-terms to the Lagrangian, the Bekenstein-Hawking entropy formula is no longer valid. A generalization of the area law has been derived by Wald [4]. The Bekenstein-Hawking area is recovered when taking the Einstein-Hilbert Lagrangian. We work with the $N = 2$ supergravity Lagrangian, which does not depend on derivatives of the Riemann tensor, and we further assume the black holes to be static and spherically symmetric. The generalized entropy formula in this case is

$$
S = 2\pi A \varepsilon_{ab} \varepsilon^{cd} \frac{\partial (e^{-1}\mathcal{L})}{\partial R_{ab}{}^{cd}} \ ,
\tag{11}
$$

where A is the (modified) area of the horizon, $\varepsilon_{01} = -\varepsilon_{10} = 1$, $\mathcal{L}$ is the Lagrangian density, and the expression is evaluated on the event horizon. In the derivative, we treat the Riemann tensor and the metric as being independent and take into account the supergravity constraints on the fields. Our derivation is similar to that of [5, 1].

For the Lagrangian (5) we get

$$
\begin{aligned}
\frac{\partial (e^{-1}\mathcal{L})}{\partial R_{ab}{}^{cd}} =& -\frac{1}{16\pi}\delta^a_c \delta^b_d + \\
& -\frac{1}{8\pi}\text{Im}\left(F_{\widehat{A}I}(F_{ef}^{-I} - \frac{1}{4}\bar{X}^I T_{ef}^-)\frac{\partial \widehat{F}^{ef-}}{\partial R_{ab}{}^{cd}} + F_{\widehat{A}\widehat{A}}\widehat{F}_{ef}^- \frac{\partial \widehat{F}^{ef-}}{\partial R_{ab}{}^{cd}} + F_{\widehat{A}}\frac{\partial \widehat{C}}{\partial R_{ab}{}^{cd}} \right) \ .
\end{aligned}
\tag{12}
$$

This expression can be simplified to[3]:

$$\frac{\partial(e^{-1}\mathcal{L})}{\partial R_{ab}{}^{cd}} = -\frac{1}{16\pi}\delta^a_c\delta^b_d +$$

$$+ \frac{1}{\pi}\text{Im}\bigg[\Big(2F_{\widehat{A}I}(F^{-I}_{pq} - \frac{1}{4}\bar{X}^I T^-_{pq})T^{mn-} - 32F_{\widehat{A}\widehat{A}}\mathcal{R}(M)^{xy}{}_{pq}T^-_{xy}T^{mn-} +$$

$$-16F_{\widehat{A}}\mathcal{R}(M)^{mn-}{}_{pq}\Big)\frac{\partial\mathcal{R}(M)_{mn}{}^{pq}}{\partial R_{ab}{}^{cd}} - F_{\widehat{A}}T^{an-}T^+_{cn}\delta^b_d\bigg] .$$

(13)

The last term is the same as in the supersymmetric case[4].

We have the relation:

$$\mathcal{R}(M)^{mn}{}_{pq} = C^{mn}{}_{pq} + D\delta^{[m}_{[p}\delta^{n]}_{q]} - 2\delta^{[m\star}_{[p}R(A)^{n]}{}_{q]} ,$$

(14)

where $C^{ab}{}_{cd}$ is the Weyl tensor. In addition, from the definition of $\mathcal{R}(M)_{ab}{}^{cd}$ (7) we get:

$$\frac{\partial\mathcal{R}(M)_{mn}{}^{pq}}{\partial R_{ab}{}^{cd}} \sim \delta^a_m\delta^b_n\delta^p_c\delta^q_d - 2\delta^{[p}_{[m}\delta^a_{n]}\delta^{q]}_c\delta^b_d ,$$

(15)

where the RHS must be constrained to have the same symmetries as the LHS, and we used $D = -\frac{1}{3}R + \dots$ due to the nonlinear multiplet constraint (9).

Substituting all expressions, we obtain the generalized entropy formula for the non-extremal R^2 case:

$$S = \frac{1}{4}A - 4A\cdot\text{Im}\Big(F_{\widehat{A}}(|T^-_{01}|^2 + 16C_{0101} + 16D)\Big) ,$$

(16)

where we have used spherical symmetry, everything is evaluated on the event horizon, and $\widehat{A} = -4(T^-_{01})^2$. This formula differs from the extremal R^2 case by the C_{0101} and D terms, where one also had

$$\widehat{A} = -256\pi A^{-1} .$$

(17)

Note that as in the extremal R^2 case, the entropy does not depend on the higher order derivatives $F_{\widehat{A}I}, F_{\widehat{A}\widehat{A}}$.

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[3]Using the identity for anti-selfdual tensors: $R^{mn-}_{pq}R_{mn}{}^{pq-} = R^{mn-}{}_{pq}R_{mn}{}^{pq}$.

[4]Using that $D_aD^cT^+_{cb} = \mathcal{D}_a\mathcal{D}^cT^+_{cb} - f_a{}^cT^+_{cb}$, for a bosonic solution. The covariant derivative string may be expanded as $\mathcal{D}_a\mathcal{D}^c = \frac{1}{2}\{\mathcal{D}_a, \mathcal{D}^c\} + \frac{1}{2}[\mathcal{D}_a, \mathcal{D}^c]$. Only the anticommutator part is dependent on the Riemann tensor, however its contribution vanishes due to the identity for (anti-)selfdual tensors: $T^{ab-}T^{c+}_b = T^{cb-}T^{a+}_b$.

4 Near-Extremal $N = 2$ Black Holes

4.1 Near-Extremal $N = 2$ Black Holes without R^2-Terms

We will start by discussing non-extremal black holes in $N = 2$ supergravity without R^2-terms [6, 7]. The metric is given by

$$ds^2 = -e^{-2U(r)}f(r)dt^2 + e^{2U(r)}(f(r)^{-1}dr^2 + r^2 d\Omega^2) \ , \tag{18}$$

where $d\Omega^2 = \sin^2\theta d\phi^2 + d\theta^2$, and

$$f(r) = 1 - \frac{\mu}{r} \ , \tag{19}$$

and $\mu \geq 0$ is a non-extremality parameter. The background is non-supersymmetric, with μ parameterizing the difference between the ADM mass and the BPS mass.

The event horizon is located at $r = \mu$ and the inner horizon at $r = 0$. Unlike the extremal black holes, the event horizon geometry is not $AdS_2 \times S^2$.

Consider the prepotential:

$$F = \frac{D_{ABC}X^A X^B X^C}{X^0} \ , \tag{20}$$

and the ansatz

$$e^{2U(r)} = e^{-K} \ , \tag{21}$$

where the Kähler potential K is

$$e^{-K} = i\Big(\bar{X}^I(\bar{z})F_I(z) - X^I(z)\bar{F}_I(\bar{z}) \Big) \ . \tag{22}$$

$F_I(z) = F_I(X(z))$ and $X^I(z)$ are related to the X^I, by

$$X^I = e^{\frac{1}{2}K}X^I(z) \ . \tag{23}$$

Consider black holes with one electric charge q_0 and p^A $(A = 1 \ldots N_V)$ magnetic charges. One introduces the boost parameters γ^A, γ_0, related to the charges by

$$
\begin{aligned}
p^A &= h^A \mu \sinh\gamma^A \cosh\gamma^A \qquad \text{(no summation)} \\
q_0 &= h_0 \mu \sinh\gamma_0 \cosh\gamma_0 \ ,
\end{aligned}
\tag{24}
$$

where h^A, h_0 are constants[5] that determine the moduli at infinity. Note that for fixed charges and non-extremality parameter, a choice of (γ^A, γ_0) is equivalent to a choice of (h^A, h_0). The extremal case is recovered in the limit $\mu \to 0; (\gamma^A, \gamma_0) \to \infty$, with the charges held fixed.

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[5]These parameters are constrained by the asymptotic flatness condition: $e^{2U(\infty)} = |h^I F_I(\infty) - h_I X^I(\infty)|^2 = 1$.

Introduce the modified charges

$$
\begin{aligned}
\tilde{p}^A &\equiv h^A \mu \sinh^2 \gamma^A = \alpha^A p^A \qquad \text{(no summation)} \\
\tilde{q}_0 &\equiv h_0 \mu \sinh^2 \gamma_0 = \alpha_0 q_0 \ ,
\end{aligned}
\tag{25}
$$

where $\alpha^A \equiv \tanh \gamma^A, \alpha_0 \equiv \tanh \gamma_0$. In the extremal case $(\alpha^A, \alpha_0) \to 1$.

In the extremal supersymmetric case, the vanishing of the gaugino variations under $N = 1$ supertransformations, implies generalized stabilization equations, also called the supersymmetric attractor mechanism [9]. These equations determine the values of the moduli on the horizon in terms of the electric and magnetic charges. In the non-extremal case the gaugino variations do not vanish. Consider an ansatz similar to the supersymmetric stabilization equations of the form

$$
\begin{aligned}
i(X^I(z) - \bar{X}^I(\bar{z})) &= \tilde{H}^I \\
i(F_I(z) - \bar{F}_I(\bar{z})) &= \tilde{H}_I \ ,
\end{aligned}
\tag{26}
$$

where $\tilde{H}^I, \tilde{H}_I$ are harmonic functions

$$
\begin{aligned}
\tilde{H}^I &= h^I + \frac{\tilde{p}^I}{r} \\
\tilde{H}_I &= h_I + \frac{\tilde{q}_I}{r} \ .
\end{aligned}
\tag{27}
$$

These equations do not exhibit an attractor behavior, since the moduli on the event horizon at $r = \mu$ depend on the moduli at infinity.

The ansatz solves the field equations for equal parameters γ^A $(A = 1 \ldots N_V)$. One can relax some of the conditions on the γ^A's by restricting the prepotential to specific choices D_{ABC}. For instance, if only one of the D_{ABC}'s (up to permutations) is nonzero, all γ^A's may be chosen independently.[6]

For the case of either equal γ^A $(A = 1, 2, 3)$ or only $D_{123} \neq 0$, the auxiliary field T_{ab}^- takes the form

$$
T_{01}^- = i T_{23}^- = \left(\frac{k_0}{\alpha_0(r + k_0)} + \frac{k^1}{\alpha^1(r + k^1)} + \frac{k^2}{\alpha^2(r + k^2)} + \frac{k^3}{\alpha^3(r + k^3)} \right) \frac{1}{r} e^{-U(r)} \ ,
\tag{28}
$$

where

$$
\begin{aligned}
k^A &\equiv \mu \sinh^2 \gamma^A = \frac{\mu(\alpha^A)^2}{1 - (\alpha^A)^2} \qquad \text{(no summation)} \\
k_0 &\equiv \mu \sinh^2 \gamma_0 = \frac{\mu(\alpha_0)^2}{1 - (\alpha_0)^2} \ .
\end{aligned}
\tag{29}
$$

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[6]Alternatively, [7] suggests a method for finding near-extremal solutions with no restrictions but only in the near-extremal regime.

Solving the stabilization equations, one obtains the moduli on the horizon in terms of the charges and the moduli at infinity. The Bekenstein-Hawking entropy takes the form

$$S = \frac{1}{4}A = 2\pi \sqrt{\left(\frac{q}{\alpha}\right)_0 D_{ABC} \left(\frac{p}{\alpha}\right)^A \left(\frac{p}{\alpha}\right)^B \left(\frac{p}{\alpha}\right)^C} \ , \tag{30}$$

where

$$\begin{aligned}
\left(\frac{p}{\alpha}\right)^A &\equiv \frac{p^A}{\alpha^A} = h^A \mu \cosh^2 \gamma^A \qquad \text{(no summation)} \\
\left(\frac{q}{\alpha}\right)_0 &\equiv \frac{q_0}{\alpha_0} = h_0 \mu \cosh^2 \gamma_0 \ .
\end{aligned} \tag{31}$$

This has the same form as the extremal entropy, with the charges (q_0, p^A) replaced by $\left(\left(\frac{q}{\alpha}\right)_0, \left(\frac{p}{\alpha}\right)^A\right)$. Note that, unlike the extremal case, the entropy depends on the values of the moduli at infinity. In addition, the non-extremal entropy has a different functional dependence on the original charges since the parameters α^A, α_0 themselves depend on the charges for given asymptotic moduli.

The near-extremal black holes are described by adding to the extremal black holes the leading terms in μ, while holding the physical charges fixed. One gets

$$\begin{aligned}
\left(\frac{p}{\alpha}\right)^A &= p^A + \frac{1}{2}h^A \mu + O(\mu^2) \\
\left(\frac{q}{\alpha}\right)_0 &= q_0 + \frac{1}{2}h_0 \mu + O(\mu^2) \ .
\end{aligned} \tag{32}$$

We see that the near-extremal Bekenstein-Hawking entropy formula has the same structure as the extremal entropy with a modification of the charges depending on the non-extremality parameter μ and the asymptotic values of the moduli h^A. In the next section we will construct a class of horizon solutions, where this structure holds with R^2-terms, as in (1) and (2).

4.2 Near-Extremal $N = 2$ Black Holes with R^2-Terms

We would like to get an explicit expression for the entropy for the near-extremal black holes with R^2-terms (16), as a function of the charges and the moduli at infinity. Consider black holes with one electric charge q_0 and p^A ($A = 1, 2, 3$) magnetic charges.

Let us introduce the dual field strength:

$$G^-_{abI} = 2i \frac{\partial(e^{-1}\mathcal{L})}{\partial F^{ab-I}} = F_{IJ} F^{-J}_{ab} + \frac{1}{4}(\bar{F}_I - F_{IJ}\bar{X}^J)T^-_{ab} + F_{\widehat{A}I}\widehat{F}^-_{ab} \ , \tag{33}$$

where we have considered only bosonic terms. Due to spherical symmetry we have

$$\begin{aligned}
F^{-I}_{23} &= -iF^{-I}_{01} \\
G^-_{23I} &= -iG^-_{01I} \ .
\end{aligned} \tag{34}$$

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The field strengths F_{01}^{-I} may be extracted from the following equations:

$$2(\mathrm{Im}F_{IJ})F_{01}^{-J} = G_{23I} - \bar{F}_{IJ}F_{23}^J + \frac{1}{2}\mathrm{Im}\Big(\big(F_I + F_{IJ}\bar{X}^J - 64F_{\widehat{A}I}(2C_{0101} - D)\big) T_{01}^- \Big) , \qquad (35)$$

where we used spherical symmetry and (14). The magnetic parts of the field strengths are obtained from Bianchi identities, which for a static spherically symmetric metric give:

$$F_{23}^I = \frac{1}{r^2}e^{-2U(r)}p^I$$
$$G_{23I} = \frac{1}{r^2}e^{-2U(r)}q_I , \qquad (36)$$

where we used $g_{\theta\theta} = g_{\phi\phi}/\sin^2\theta = r^2e^{2U(r)}$. For our choice of charges, and the complex-valued form of our ansatz (41) we get

$$F_{01}^{-0} = \frac{1}{2F_{00}}\Big(iG_{230} - iF_{0A}F_{23}^A + \frac{1}{2}\big(F_0 + F_{0I}\bar{X}^I - 64F_{\widehat{A}0}(2C_{0101} - D)\big) T_{01}^-\Big)$$
$$F_{01}^{-A} = \frac{i}{2}F_{23}^A . \qquad (37)$$

The auxiliary field D may be determined by the constraint on the nonlinear multiplet (9):

$$D = \mathcal{D}^a V_a - \frac{1}{3}R - \frac{1}{2}V^a V_a - \frac{1}{4}M_{ij}\bar{M}^{ij} + D^a\Phi^i{}_\alpha D_a\Phi^\alpha{}_i . \qquad (38)$$

This means that we must make the above substitution also for D appearing in the hatted fields (6) in the Lagrangian (5). We assume

$$V_a = 0$$
$$M_{ij} = 0$$
$$\Phi^i{}_\alpha = \delta^i_\alpha , \qquad (39)$$

where the later equation is the V-gauge (10). The equation of motion for V_a must be shown to be satisfied. The equations of motion for M_{ij} and $\Phi^i{}_\alpha$ are trivially satisfied by the above assumption, where for the latter we assume a vanishing $SU(2)$ connection as we shall consider later. We therefore remain with the constraint:

$$D = -\frac{1}{3}R . \qquad (40)$$

The area of the horizon A, the Weyl tensor C_{0101}, and the Ricci scalar R are all calculated from the metric. It remains to find solutions for the metric, the moduli $X^I(z)$, and the auxiliary field T_{01}^-. In addition, for solving the equations of motion, we will need solutions for the $U(1)$

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connection A_a and the $SU(2)$ connection $\mathcal{V}_a{}^i{}_j$, which in the supersymmetric case could be taken as zero. We will make an ansatz for the solution on the horizon, which is an extension of both the extremal case with R^2-terms (see [1]) and the non-extremal case without R^2-terms. One may consider the ansatz of the non-extremal case for the metric (18), (21), (19), the modified stabilization equations (26) which give the moduli, and the auxiliary field (28), with the R^2 prepotential (3). However this proves to be insufficient, and since we will consider a near-extremal solution, we introduce linear μ-corrections to the fields.

Our ansatz is

$$
\begin{aligned}
F &= \frac{D_{ABC}X^A X^B X^C}{X^0} + \frac{D_A X^A}{X^0}\widehat{A} \\
ds^2 &= -e^{-2U(r)}f(r)dt^2 + e^{2U(r)}(f(r)^{-1}dr^2 + r^2 d\Omega^2) \\
e^{2U(r)} &= e^{-K}(1 + \mu\beta_U) \\
f(r) &= \left(1 - \frac{\mu}{r}\right)(1 + \mu\beta_f) \\
X^A(z) &= -\frac{i}{2}x^A(1 + \mu\beta_A) \\
X^0(z) &= \frac{1}{2}\sqrt{\frac{D_{ABC}x^A x^B x^C - 4D_A x^A \widehat{A}(z)}{x_0}}(1 + \mu\beta_0) \\
T_{01}^- &= iT_{23}^- = \left(\frac{k_0}{\alpha_0(r + k_0)} + \frac{k^1}{\alpha^1(r + k^1)} + \frac{k^2}{\alpha^2(r + k^2)} + \frac{k^3}{\alpha^3(r + k^3)}\right)\frac{1}{r}e^{\frac{1}{2}K}(1 + \mu\beta_T) ,
\end{aligned}
$$
$$(41)$$

where

$$
\begin{aligned}
x^A &\equiv \frac{\alpha^A p^A}{k^A} + \frac{\alpha^A p^A}{r} \qquad \text{(no summation)} \\
x_0 &\equiv \frac{\alpha_0 q_0}{k_0} + \frac{\alpha_0 q_0}{r} ,
\end{aligned}
$$
$$(42)$$

and

$$
\begin{aligned}
\widehat{A}(z) &= e^{-K}\widehat{A} = -4e^{-K}(T_{01}^-)^2 = \\
&= -4\left(\frac{k_0}{\alpha_0(r + k_0)} + \frac{k^1}{\alpha^1(r + k^1)} + \frac{k^2}{\alpha^2(r + k^2)} + \frac{k^3}{\alpha^3(r + k^3)}\right)^2 \frac{1}{r^2}(1 + \mu\beta_T)^2 .
\end{aligned}
$$
$$(43)$$

$\beta_U, \beta_f, \beta_1, \beta_2, \beta_3, \beta_0, \beta_T$ are constants satisfying $\mu|\beta| \ll 1$, and e^{-K} also contains β's. Note that besides the explicit μ-corrections above, some of the fields will also have implicit μ dependence via e^{-K} and $A(z)$.

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In addition we assume

$$A_a = 0$$
$$\mathcal{V}_a{}^i{}_j = 0 .$$

(44)

The equation of motion for the $SU(2)$ connection is always satisfied by the vanishing $SU(2)$ connection, for a bosonic background and with our choice of V-gauge (also assuming no hypermultiplet scalars). This is because the $SU(2)$ connection and its derivatives, appear then in the Lagrangian (5) always in at least a quadratic form.[7] The vanishing of the $SU(2)$ connection implies also $Y_{ij}^I = 0$ [10].

For our ansatz to constitute a solution, it must satisfy the equations of motion on the horizon for the metric, the moduli $X^I(z)$, the auxiliary field T_{01}^-, the $U(1)$ connection A_a, and the nonlinear multiplet field V_a. In general the above ansatz is not a solution to the equations of motion. However, we have found that it may constitute a near-extremal horizon solution if we require equal boost parameters and $D_{ABC}p^Ap^Bp^C = 0$. The latter condition on the charges implies the vanishing of the classical horizon area in the extremal limit.[8]

In the near-extremal regime we linearize the algebraic equations of motion (after substituting the ansatz) in the small expansion parameter $\mu \ll (2k^A, 2k_0)$. Recall that the boost parameters α^A, α_0 depend on μ with constant k^A, k_0:

$$\alpha^A = \sqrt{\frac{k^A}{k^A + \mu}} \qquad \text{(no summation)}$$
$$\alpha_0 = \sqrt{\frac{k_0}{k_0 + \mu}} .$$

(45)

The expansion must be done after taking the horizon limit $r \to \mu$, since for a small but finite μ we want to have two topologically distinct horizons in the metric function. Next, we choose equal boost parameters: $\alpha_0 = \alpha^1 = \alpha^2 = \alpha^3$. Without R^2-terms, this would be the non-extremal version of the double-extremal black hole. Denote: $k \equiv k_0 = k^1 = k^2 = k^3$. Finally, $D_{ABC}p^Ap^Bp^C = 0$ would imply a vanishing of the classical horizon area for the extremal R-level case (i.e. without R^2-terms) [11].

---

[7]More generally, the $SU(2)$ field equations are automatically satisfied since the solution is a singlet under the $SU(2)$ symmetry.

[8]Note added: In a later paper we construct non-extremal R^2 solutions in all space with $D_{ABC}p^Ap^Bp^C \neq 0$, by using the large charge approximation. The entropy of these solutions exhibits the same charge replacement property as seen in the R-level solutions. [arXiv:0902.3799]

Under these three restrictions our ansatz (41) solves the field equations, with the β's for some simplified cases given in appendix A. In appendix B we comment on the derivation of the metric field equations.

For the above ansatz, the area of the horizon, the Ricci scalar and the Weyl tensor on the horizon read

$$
\begin{aligned}
A &= 4\pi\mu^2 e^{-K(r=\mu)} \\
R &= \frac{\mu\beta_f}{8\sqrt{q_0 D_A p^A}} + O(\mu^2) \\
C_{0101} &= -\frac{\mu\beta_f}{48\sqrt{q_0 D_A p^A}} + O(\mu^2) \ .
\end{aligned}
\tag{46}
$$

Substituting the solution in the generalized entropy formula for the near-extremal R^2 case (16) yields the explicit entropy:

$$
S = 32\pi\sqrt{q_0 D_A p^A} + O(\mu^2),
\tag{47}
$$

where we must choose the signs of the charges such that the result is real. It turns out that the linear μ-terms which appear in the Bekenstein-Hawking entropy and in the Wald correction to the entropy, exactly cancel. Thus to this order of approximation, the entropy does not depend on μ, nor on the asymptotic moduli at infinity, and is the same as in the extremal case. This does not exhibit the same shift of charges as in the transition from the R-level extremal entropy to the near-extremal entropy. It would be interesting to compare the obtained expression for the entropy, to a corresponding microscopic statistical entropy, which is currently unknown.

We have considered only the tree-level α' F-terms. For this we require that the large volume approximation is valid near the horizon, by imposing: $|q_0| \gg |p^3|$. We may further take the magnetic charges to be large, in order to damp out any R^4 or higher D-term corrections. However, we cannot rule out other contributions to the field solutions and entropy coming from D-terms at the R^2-level.

The Hawking temperature for our static spherically symmetric black hole is given by

$$
T = -\frac{\partial_r g_{tt}}{4\pi\sqrt{-g_{tt}g_{rr}}}\bigg|_{horizon} = \frac{\mu}{64\pi\sqrt{q_0 D_A p^A}} + O(\mu^3) \ .
\tag{48}
$$

Note that the quadratic term in μ vanishes. Also, since the entropy has a vanishing linear term in μ, the first law of thermodynamics implies that the mass has vanishing linear and quadratic terms in μ.[9]

---

[9]$T = \frac{\partial M}{\partial S} = \frac{\partial M(q_I, p^I, h_I, h^I, \mu)}{\partial \mu}\left(\frac{\partial S(q_I, p^I, h_I, h^I, \mu)}{\partial \mu}\right)^{-1}$, where M is the mass, q_I, p^I, h_I, h^I are held fixed, and assuming $\partial S/\partial\mu \neq 0$.

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Since the solutions have been constructed only on the horizon, and without the supersymmetry property, one still needs to analyze whether an interpolating solution exists which smoothly connects the horizon to asymptotically flat space. This is a prerequisite for the existence of a corresponding black hole and for the validity of the Wald entropy formula (11).

Acknowledgements

A   Solutions of the Field Equations

Following are the explicit solutions for the β's of (41) which satisfy the equations of motion, under the discussed restrictions. The free real parameter β_U, represents a gauge freedom that should be fixed by the normalization of the interpolating solution at infinity. Our calculations were done using Maple with GRTensor.

(i) For the case $D_{113} = D_{133} = D_{223} = D_{233} = D_{123} = D_1 = D_2 = 0$:

$$
\begin{aligned}
\beta_f &= -\frac{3D_{333}p^3 p^3}{256k D_3} \\
2\beta_0 &= \beta_f \\
2\beta_1 = 2\beta_2 &= \beta_f - \beta_U \\
2\beta_3 &= \beta_f - \frac{1}{k} - \beta_U \\
2\beta_T &= \frac{1}{k} - \beta_U \ .
\end{aligned}
\tag{49}
$$

(ii) For the case $D_{112} = D_{122} = D_{223} = D_{233} = D_{222} = D_{123} = D_2 = 0$:

$$
\begin{aligned}
\beta_f &= \frac{3X^2}{256kY} \\
2\beta_0 &= \beta_f \\
2\beta_1 &= \beta_f - \frac{D_3 p^3 X}{kY} - \frac{1}{k} - \beta_U \\
2\beta_2 &= \text{unconstrained} \\
2\beta_3 &= \beta_f + \frac{D_1 p^1 X}{kY} - \frac{1}{k} - \beta_U \\
2\beta_T &= \frac{1}{k} - \beta_U \ ,
\end{aligned}
\tag{50}
$$

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where

$$
\begin{aligned}
X &= \frac{1}{2} \left(D_{333} p^3 p^3 p^3 + D_{133} p^1 p^3 p^3 - D_{113} p^1 p^1 p^3 - D_{111} p^1 p^1 p^1 \right) \\
Y &= \left(D_1 D_{333} p^3 p^3 + D_1 D_{133} p^1 p^3 + D_3 D_{113} p^1 p^3 + D_3 D_{111} p^1 p^1 \right) p^1 p^3 .
\end{aligned}
$$
(51)

Note that here it is assumed that $Y \neq 0$ and

$$
D_{333} p^3 p^3 p^3 + 3 D_{133} p^1 p^3 p^3 + 3 D_{113} p^1 p^1 p^3 + D_{111} p^1 p^1 p^1 = 0 .
$$
(52)

(iii) For the case $D_{112} = D_{122} = D_{113} = D_{133} = D_{223} = D_{233} = D_{111} = D_{222} = D_{333} = 1$, $D_{123} = -\frac{7}{2}$, and $p^1 = p^2 = p^3$:

$$
\begin{aligned}
\beta_f &= 0 \\
2\beta_0 &= 0 \\
2\beta_1 = 2\beta_2 = 2\beta_3 &= -\frac{1}{k} - \beta_U \\
2\beta_T &= \frac{1}{k} - \beta_U .
\end{aligned}
$$
(53)

B   Derivation of the Metric Field Equations

In order to simplify the derivation of the equations of motion, we write the Lagrangian in a form which is explicit in the scalar degrees of freedom. There are some subtleties regarding the degrees of freedom of the metric. Here we will identify these degrees of freedom and how they should be accounted for in the computation.

Let $\mathcal{L}(\psi, \partial_\mu \psi, \partial_\mu \partial_\nu \psi)$ be a Lagrangian density depending on the scalar field ψ and its first and second space-time derivatives. The equation of motion for ψ is given by the Euler-Lagrange equation:

$$
\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu \psi)} \right) = 0 .
$$
(54)

In our case, the action contains curvature tensors which are built from second order derivatives. Thus we need to take the full second order variation. Alternatively, one may integrate the action by parts, and take the usual first order variation.

We assume a static and spherically symmetric metric. A general form of such a metric is

$$
ds^2 = -e^{-2U_1(r)} dt^2 + e^{2U_2(r)} dr^2 + e^{2U_3(r)} r^2 d\Omega^2 .
$$
(55)

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Correspondingly, we will get three equations of motion for $U_1(r), U_2(r), U_3(r)$. Any metric has only two real degrees of freedom: $16 - 6$ (symmetric components) - 4 (Bianchi identities) - 4 (coordinate redefinitions) = 2. So our static and spherically symmetric metric really contains only two independent r-function degrees of freedom. Thus one of the three equations of motion will be redundant.

We will explain why the above policy is nevertheless advantageous. One may set e.g. $U_2(r) = U_1(r)$ by a redefinition of the r-coordinate. This choice of "gauge" may result in a trivial equation of motion for $U_1(r)$, leaving us with only one independent equation of motion. The missing equation of motion has to be obtained from the requirement that the action is invariant under the choice of gauge. I.e. the variation of the non-gauged-fixed action with respect to $U_2(r)$ must vanish. This is also known as a Hamiltonian constraint. However, this is just the original equation of motion for $U_2(r)$ that we threw away by the gauge fixing. Thus we will simply retain all three degrees of freedom in the metric, which will give two independent equations of motion after gauge fixing.

One may now be concerned about other gauge fixings implicit in the choice of coordinates of (55), e.g. vanishing off-diagonal components or $g_{\theta\theta} = g_{\phi\phi}/sin^2\theta$. However, our metric is the "maximally general" metric preserving the assumed isometries of the solution, namely staticity and spherical symmetry [12]. For such a solution, the equations of motion corresponding to the trivial metric components would be automatically satisfied and would not yield new constraints.

In order to be consistent with the notation of our solution (41), we will actually use the metric:

$$ds^2 = -e^{-2U_1(r)}f(r)dt^2 + e^{2U_2(r)}f(r)^{-1}dr^2 + e^{2U_3(r)}r^2d\Omega^2 \ , \tag{56}$$

where $f(r)$ is given. The solution to the equations of motion is given by

$$U_1(r) = U_2(r) = U_3(r) = U(r) \ , \tag{57}$$

where $U(r)$ is given. When deriving the equations of motion, we must retain the separate degrees of freedom of the metric.

The fields F_{ab}^{-I}, T_{ab}^{-} in our solution, are given as the anti-selfdual parts written with tangent space indices. In this form, these fields contain metric components, while the metric-independent fields are $F_{\mu\nu}^I, T_{\mu\nu}$. Let us denote by $F_{01}^{-I}(r), T_{01}^{-I}(r)$ the $(0,1)$ components of these fields as given in our solution (37), (41), before we explicitly introduced the separate metric degrees of freedom. When these fields appear in the Lagrangian explicitly (including via the hatted fields (6)), they

should be rewritten as

$$
\begin{aligned}
F_{01}^{-A} &= iF_{23}^{-A} = e^{2U(r)-2U_3(r)}F_{01}^{-A}(r) \\
F_{01}^{-0} &= iF_{23}^{-0} = e^{U_1(r)-U_2(r)}F_{01}^{-0}(r) \\
T_{01}^{-} &= iT_{23}^{-} = e^{U_1(r)-U_2(r)}T_{01}^{-}(r) \ .
\end{aligned}
\tag{58}
$$

Alternatively, one may work with the $F_{\mu\nu}^{I}, T_{\mu\nu}$ form and put appropriate projection operators in the Lagrangian.

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